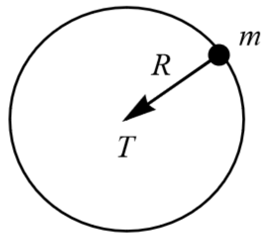
Problem 1.41

An astronaut in gravity-free space is twirling a mass m on the end of a string of length R in a circle, with constant angular velocity ω. Write down Newton's second law (1.48) in polar coordinates and find the tension in the string.

Solution

Start by drawing a free-body diagram of the mass. There's only a tensile force acting on the mass, which points toward the center of the circle.



Newton's second law states that the sum of the forces on the mass is equal to its mass times acceleration.

$$\sum \mathbf{F} = m\mathbf{a} \quad \Rightarrow \quad \begin{cases} \sum F_r = ma_r \\ \sum F_\phi = ma_\phi \\ \sum F_z = ma_z \end{cases}$$

The tension acts in the negative r-direction.

$$\begin{cases} -T = ma_r \\ 0 = ma_\phi \\ 0 = ma_z \end{cases}$$

Divide both sides of each equation by m.

$$\begin{cases} -\dfrac{T}{m} = a_r \\ 0 = a_\phi \\ 0 = a_z \end{cases}$$

Substitute the formulas for acceleration in cylindrical coordinates.

$$\begin{cases} \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\phi}{dt}\right)^2 = -\dfrac{T}{m} \\ r\dfrac{d^2\phi}{dt^2} + 2\dfrac{dr}{dt}\dfrac{d\phi}{dt} = 0 \\ \dfrac{d^2z}{dt^2} = 0 \end{cases}$$

Because the mass moves in a circle, $r = R$, $dr/dt = 0$, and $d^2r/dt^2 = 0$.

$$\begin{cases} 0 - R\left(\frac{d\phi}{dt}\right)^2 = -\frac{T}{m} \\ R\frac{d^2\phi}{dt^2} + 2(0)\left(\frac{d\phi}{dt}\right) = 0 \\ \frac{d^2z}{dt^2} = 0 \end{cases}$$

And since the mass moves with a constant angular velocity, $d\phi/dt = \omega$ and $d^2\phi/dt^2 = 0$.

$$\begin{cases} 0 - R(\omega)^2 = -\frac{T}{m} \\ R(0) + 2(0)(\omega) = 0 \\ \frac{d^2z}{dt^2} = 0 \end{cases}$$

Therefore, multiplying both sides of the first equation by $-m$,

$$T = mR\omega^2.$$